The definition of the Hilbert transform of a real signal $x(t)$ should be

$$h(t) = H\{x(t)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\mathrm{d}\tau\,. \qquad (1)$$

There is no minus sign in the integral.

Proof:

By change of variable, $t-\tau = y$, we have $\tau = t - y$, and $\mathrm{d}\tau = -\mathrm{d}y$. The integral limits of (1) become from ∞ to $-\infty$. Then, the above integral can be rewritten as

$$h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\mathrm{d}\tau = -\frac{1}{\pi}\int_{\infty}^{-\infty}\frac{x(t-y)}{y}\mathrm{d}y = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(t-y)}{y}\mathrm{d}y = \frac{1}{\pi t} * x(t)\,. \qquad (2)$$

Using the convolution property of the Fourier transform, its Fourier transform is

$$h(t) \Leftrightarrow H(\omega) = \mathscr{F}\left\{\frac{1}{\pi t}\right\}\bullet\mathscr{F}\{x(t)\}\,. \qquad (3)$$

$$\mathscr{F}\left\{\frac{1}{\pi t}\right\} = \int_{-\infty}^{\infty}\frac{1}{\pi t}e^{-j\omega t}\mathrm{d}t = \frac{1}{\pi}\left[\int_{-\infty}^{\infty}\frac{\cos(\omega t)}{t}\mathrm{d}t - j\int_{-\infty}^{\infty}\frac{\sin(\omega t)}{t}\mathrm{d}t\right]. \qquad (4)$$

Since $\cos(\omega t)/t$ is an odd function, the first integral in (4) vanishes. Considering $\sin(\omega t)/t$ is an even function, and from a integral formula table,

$$\int_{0}^{\infty}\frac{\sin\omega t}{t}\mathrm{d}t = \begin{cases}\pi/2, & \omega > 0\\ 0, & \omega = 0\\ -\pi/2, & \omega < 0\end{cases},$$

(4) becomes $\mathscr{F}\left\{\frac{1}{\pi t}\right\} = -j\,\mathrm{sgn}(\omega)$. Then the Fourier transform of the Hilbert transform $h(t)$ is

$$H(\omega) = -j\,\mathrm{sgn}(\omega)X(\omega)\,. \qquad (5)$$

Q.E.D